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A New “Null-Plane” Quantum Poincaré Algebra

A. Ballesteros¹, F.J. Herranz¹, M.A. del Olmo² and M. Santander²

¹ *Departamento de Física Aplicada, Universidad de Burgos
E-09003, Burgos, Spain*

² *Departamento de Física Teórica, Universidad de Valladolid
E-47011, Valladolid, Spain*

Abstract

A new quantum deformation, which we call null-plane, of the (3+1) Poincaré algebra is obtained. The algebraic properties of the classical null-plane description are generalized to this quantum deformation. In particular, the classical isotopy subalgebra of the null-plane is deformed into a Hopf subalgebra, and deformed spin operators having classical commutation rules can be defined. Quantum Hamiltonian, mass and position operators are studied, and the null-plane evolution is expressed in terms of a deformed Schrödinger equation.

1 Introduction

A central problem of a Hamiltonian formulation of quantum relativistic systems consists in the obtention of a set of dynamically independent variables in terms of the generators of the (3+1) Poincaré group. For the null-plane evolution [1] this has been thoroughly studied, and in this case it is well known that the information provided by the Poincaré invariance into a kinematical and a dynamical part. To achieve this by preserving causality, one can consider that the initial state of the system is defined on a light-like plane Π_n^τ (a hypersurface of points x of the Minkowski space such that $n \cdot x = \tau$, where n is a light-like vector and τ a real constant). The stability group S_+ of Π_n^τ gives the kinematics inside the null-plane, and the dynamics is obtained by means of the remaining generators (the Hamiltonians) that map Π_n^τ into some other surface and describe the evolution of the system. A brief summary of this construction is given in Section 2 for the case $n = (\frac{1}{2}, 0, 0, \frac{1}{2})$ (see [2] for a general review).

In this letter we present a new quantum deformation of the Poincaré algebra such that the classical null-plane isotopy subalgebra S_+ is promoted into a Hopf subalgebra under deformation. The classical null-plane separation between kinematics and dynamics is therefore preserved in the quantum case. So, the study of deformations of the classical models built within this framework (for instance, the infinite momentum frame approach [3], gauge field theory quantized on null-planes [4] and applications in Hadron spectroscopy [5, 6]) could be physically interesting.

We also recall that previous Hopf algebra deformations of the (3+1) Poincaré algebra [7, 8, 9] have been obtained within the purely kinematical framework where their classical counterparts live. However, the quantum algebra here introduced induces a (dynamical) deformation of the Schrödinger equation governing the evolution of the wave functions with support on the null-plane.

From an algebraic point of view the (3+1) Poincaré deformation introduced in Section 3 is a quantization of a coboundary Lie bialgebra generated by a (non-standard) r -matrix that fulfills the classical Yang–Baxter equation (CYBE). Starting from a similar (2+1) null-plane deformation, which was introduced in [10], this new (3+1) quantum Poincaré algebra is obtained by applying a deformation embedding method (fully described in [9, 11] for the standard deformation). It is worth remarking that the deformation of the most relevant structures in the classical null-plane description is straightforward: a centrally extended (2+1) Galilean subalgebra of $\mathcal{P}(3+1)$ is transformed into a quantum algebra under deformation, the quantum central elements can be used to define the deformed mass and Pauli–Lubanski operators, and besides the latter induces a natural definition for the quantum spin operators, whose commutation rules are shown to be the classical ones. Quantum analogues of reduced Hamiltonians, position operators and the deformed uncertainty relations linked to them are studied in Section 4. Some relevant remarks close the paper.

2 Classical Poincaré algebra in a null-plane basis

Hereafter we shall consider the null-plane “orthogonal” to the light-like vector $n = (\frac{1}{2}, 0, 0, \frac{1}{2})$ as our initial surface [2]. A coordinate system well adapted to null-planes Π_n^τ is given by

$$x^- = n \cdot x = \frac{1}{2}(x^0 - x^3) = \tau, \quad x^+ = x^0 + x^3, \quad x_T = (x^1, x^2). \quad (2.1)$$

Hence, a point x contained in the null-plane Π_n^τ will be labelled by the coordinates (x^+, x^1, x^2) . The remaining coordinate x^- plays the role of a time. A particular null-plane is Π_n^0 , ($\tau = 0$), i.e., $x^- = n \cdot x = 0$; this plane is mapped onto itself by the boosts generated by K_3 , which leave the transverse coordinates $x_T = (x^1, x^2)$ unchanged and $e^{\chi K_3}$ maps x_+ on $e^\chi x_+$. The Poincaré algebra generators can be classified according to their transformation property with respect to K_3 ; any generator A obeying

$$[K_3, A] = \gamma A, \quad (2.2)$$

is referred to as an operator of “goodness” γ . The ten generators of $\mathcal{P}(3+1)$ (adapted to the null-plane coordinates in terms of the usual physical basis) can be listed in relation to their goodness in the following way:

$$\begin{aligned} \gamma = +1 : \quad & E_1 = \frac{1}{2}(K_1 + J_2), \quad E_2 = \frac{1}{2}(K_2 - J_1), \quad P_+ = \frac{1}{2}(P_0 + P_3); \\ \gamma = 0 : \quad & K_3, \quad J_3, \quad P_1, \quad P_2; \\ \gamma = -1 : \quad & F_1 = \frac{1}{2}(K_1 - J_2), \quad F_2 = \frac{1}{2}(K_2 + J_1), \quad P_- = \frac{1}{2}(P_0 - P_3). \end{aligned} \quad (2.3)$$

The non-vanishing Lie brackets of $\mathcal{P}(3+1)$ are ($i, j = 1, 2$):

$$\begin{aligned} [K_3, P_+] &= P_+, \quad [K_3, P_-] = -P_-, \quad [K_3, E_i] = E_i, \quad [K_3, F_i] = -F_i, \\ [J_3, P_i] &= -\varepsilon_{ij3} P_j, \quad [J_3, E_i] = -\varepsilon_{ij3} E_j, \quad [J_3, F_i] = -\varepsilon_{ij3} F_j, \\ [E_i, P_j] &= \delta_{ij} P_+, \quad [F_i, P_j] = \delta_{ij} P_-, \quad [P_+, F_i] = -P_i, \\ [E_i, F_j] &= \delta_{ij} K_3 + \varepsilon_{ij3} J_3, \quad [P_-, E_i] = -P_i. \end{aligned} \quad (2.4)$$

The set of all Poincaré generators with the same goodness γ spans a subgroup G_γ (in particular, the subgroups $G_{\pm 1}$ are Abelian). Moreover, there exist two seven-parametrical Poincaré subgroups with a semidirect product structure: $S_\pm = G_0 \odot G_{\pm 1}$. The stability group of the plane Π_n^0 coincides with the group S_+ . The remaining three generators, which originate G_{-1} , act on Π_n^0 as follows: the generator P_- translates Π_n^0 into Π_n^τ , while the generators F_i rotate it around the surface of the light-cone $x^2 = 0$. Therefore, if $x^- = \tau$ is considered as an evolution parameter, then P_- and F_i describe the dynamical evolution from the null-plane $x^- = 0$; they are called Hamiltonians.

The two Casimirs of $\mathcal{P}(3+1)$ are given by:

$$M^2 = 2P_- P_+ - P_1^2 - P_2^2, \quad (2.5)$$

$$W^2 = W_{13}^2 + W_{23}^2 + W_+ W_- + W_- W_+, \quad (2.6)$$

where

$$\begin{aligned} W_{13} &= K_3 P_1 + E_1 P_- - F_1 P_+, & W_{23} &= K_3 P_2 + E_2 P_- - F_2 P_+, \\ W_+ &= E_1 P_2 - E_2 P_1 + J_3 P_+, & W_- &= F_1 P_2 - F_2 P_1 + J_3 P_-. \end{aligned} \quad (2.7)$$

The second-order Casimir M^2 is the square of the mass operator (M) and the fourth-order one W^2 is the square of the Pauli-Lubanski operator. Lie brackets involving the components W_i and the null-plane generators of $\mathcal{P}(3+1)$ are given in Appendix A.

3 Quantum Poincaré algebra in a null-plane basis

In [9], the standard deformation of $\mathcal{P}(3+1)$ has been derived by using the so-called *deformation embedding method*, a kind of quantum analogue of the classical embedding of the $\mathcal{P}(2+1)$ algebra into $\mathcal{P}(3+1)$ (see also Ref. [11] for arbitrary dimension). In this way, the $(2+1)$ dimensional deformation of a given (affine) algebra turns out to be the cornerstone for the obtention of its higher dimensional generalizations. Since a null-plane quantum $\mathcal{P}(2+1)$ algebra has been given in [10], the same procedure can be now applied as follows:

1) Let us look for classical subalgebras of $\mathcal{P}(3+1)$ isomorphic to the null-plane algebra $\mathcal{P}(2+1)$: there exist two of them. Namely,

$$\Pi_{13} = \langle K_3, E_1, F_1, P_1, P_+, P_- \rangle, \quad \Pi_{23} = \langle K_3, E_2, F_2, P_2, P_+, P_- \rangle. \quad (3.1)$$

2) We assume that $U_z \Pi_{13}$ and $U_z \Pi_{23}$, endowed with the null-plane $(2+1)$ deformation of [10] are “restrictions” [11] of the quantization for $U_z \mathcal{P}(3+1)$.

3) Consider the simplest Ansatz for the coproduct and commutation rules of $U_z \mathcal{P}(3+1)$ consistent with these restrictions.

4) Check the selfconsistency of the structure obtained in 3) by imposing that the coproduct is a Hopf algebra homomorphism.

The final result is the following null-plane deformation of $U_z \mathcal{P}(3+1)$, with coproduct given by:

$$\begin{aligned} \Delta(X) &= 1 \otimes X + X \otimes 1, & \text{for } X &\in \{P_+, E_1, E_2, J_3\}, \\ \Delta(Y) &= e^{-zP_+} \otimes Y + Y \otimes e^{zP_+}, & \text{for } Y &\in \{P_-, P_1, P_2\}, \\ \Delta(F_1) &= e^{-zP_+} \otimes F_1 + F_1 \otimes e^{zP_+} + ze^{-zP_+} E_1 \otimes P_- - zP_- \otimes E_1 e^{zP_+} \\ &\quad + ze^{-zP_+} J_3 \otimes P_2 - zP_2 \otimes J_3 e^{zP_+}, \\ \Delta(F_2) &= e^{-zP_+} \otimes F_2 + F_2 \otimes e^{zP_+} + ze^{-zP_+} E_2 \otimes P_- - zP_- \otimes E_2 e^{zP_+} \\ &\quad + ze^{-zP_+} J_3 \otimes P_1 - zP_1 \otimes J_3 e^{zP_+}, \\ \Delta(K_3) &= e^{-zP_+} \otimes K_3 + K_3 \otimes e^{zP_+} + ze^{-zP_+} E_1 \otimes P_1 - zP_1 \otimes E_1 e^{zP_+} \\ &\quad + ze^{-zP_+} E_2 \otimes P_2 - zP_2 \otimes E_2 e^{zP_+}; \end{aligned} \quad (3.2)$$

commutation relations:

$$[K_3, P_+] = \frac{\sinh zP_+}{z}, \quad [K_3, P_-] = -P_- \cosh zP_+, \quad [K_3, E_i] = E_i \cosh zP_+,$$

$$\begin{aligned}
[K_3, F_1] &= -F_1 \cosh zP_+ + zE_1P_- \sinh zP_+ - z^2P_2W_+^q, \\
[K_3, F_2] &= -F_2 \cosh zP_+ + zE_2P_- \sinh zP_+ + z^2P_1W_+^q, \\
[J_3, P_i] &= -\varepsilon_{ij3}P_j, \quad [J_3, E_i] = -\varepsilon_{ij3}E_j, \quad [J_3, F_i] = -\varepsilon_{ij3}F_j, \\
[E_i, P_j] &= \delta_{ij} \frac{\sinh zP_+}{z}, \quad [F_i, P_j] = \delta_{ij}P_- \cosh zP_+, \\
[E_i, F_j] &= \delta_{ij}K_3 + \varepsilon_{ij3}J_3 \cosh zP_+, \quad [P_+, F_i] = -P_i, \\
[F_1, F_2] &= z^2P_-W_+^q + zP_-J_3 \sinh zP_+, \quad [P_-, E_i] = -P_i.
\end{aligned} \tag{3. 3}$$

(the quantum component W_+^q of the Pauli–Lubanski vector is defined below (3. 9)); counit and antipode:

$$\epsilon(X) = 0; \quad \gamma(X) = -e^{3zP_+} X e^{-3zP_+}, \quad \text{for } X \in \{P_\pm, P_i, E_i, F_i, J_3\}. \tag{3. 4}$$

As a byproduct of the embedding method, the underlying coboundary Lie bialgebra is also obtained. Explicitly, the classical r -matrix

$$r = 2z(K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2), \tag{3. 5}$$

is a skew-solution of the CYBE (it can be identified with one appearing in Zakrzewski's classification [12]), and generates the cocommutator $\delta(X) = [1 \otimes X + X \otimes 1, r]$ that provides the first order term in z within the coproduct (3. 2):

$$\begin{aligned}
\delta(X) &= 0, \quad \text{for } X \in \{P_+, E_1, E_2, J_3\}, \\
\delta(Y) &= 2zY \wedge P_+, \quad \text{for } Y \in \{P_-, P_1, P_2\}, \\
\delta(F_1) &= 2z(F_1 \wedge P_+ + E_1 \wedge P_- + J_3 \wedge P_2), \\
\delta(F_2) &= 2z(F_2 \wedge P_+ + E_2 \wedge P_- + J_3 \wedge P_1), \\
\delta(K_3) &= 2z(K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2).
\end{aligned} \tag{3. 6}$$

The mass (2. 5) and the Pauli-Lubanski (2. 6) operators can be deformed as follows:

$$M_q^2 = 2P_- \frac{\sinh zP_+}{z} - P_1^2 - P_2^2, \tag{3. 7}$$

$$W_q^2 = (W_{13}^q)^2 + (W_{23}^q)^2 + \cosh zP_+ \{W_+^q W_-^q + W_-^q W_+^q\} - z^2 M_q^2 (W_+^q)^2, \tag{3. 8}$$

where

$$\begin{aligned}
W_{13}^q &= K_3 P_1 + E_1 P_- \cosh zP_+ - F_1 \frac{\sinh zP_+}{z}, \\
W_{23}^q &= K_3 P_2 + E_2 P_- \cosh zP_+ - F_2 \frac{\sinh zP_+}{z}, \\
W_+^q &= E_1 P_2 - E_2 P_1 + J_3 \frac{\sinh zP_+}{z}, \\
W_-^q &= F_1 P_2 - F_2 P_1 + J_3 P_- \cosh zP_+.
\end{aligned} \tag{3. 9}$$

The commutation rules between the components W_i^q and the Hopf algebra generators are given in Appendix A; they show that W_q^2 belongs to the center of $U_z \mathcal{P}(3+1)$ as M_q^2 .

It is interesting to analyse now the deformed algebras given by the generators of the classical subgroups G_γ , $\gamma = -1, 0, +1$ and S_\pm . It is a rather remarkable fact that the generators of the null-plane stability group, S_+ , determine a deformed Hopf subalgebra of $U_z\mathcal{P}(3+1)$ (that we shall denote by U_zS_+), while the Lie subalgebra generating G_{+1} gives a trivial (i.e., non deformed) Hopf subalgebra under deformation. On the other hand, G_0 , G_{-1} and S_- do not originate Hopf subalgebras. Moreover, although the classical group G_{-1} is Abelian, this is not the case in the quantum version.

There is another physically interesting deformed Hopf subalgebra in $U_z\mathcal{P}(3+1)$: the set of generators $\langle J_3, E_1, E_2, P_1, P_2, P_+, P_- \rangle$ generates a quantum deformation, $U_z\tilde{\mathcal{G}}(2+1)$, of the $(2+1)$ extended Galilean algebra. The generator P_+ is central, and the structure of $U_z\tilde{\mathcal{G}}(2+1)$ can be studied by considering a new physical basis $\{\tilde{H}, \tilde{P}_1, \tilde{P}_2, \tilde{K}_1, \tilde{K}_2, \tilde{J}\}$ linked, as usual, to the time translation, space translations, boosts and space rotation, respectively. For the Galilean case we introduce a central element \tilde{M} which can be identified with the mass of a free particle. Explicitly, the relations between the null-plane and the usual kinematical bases for this subalgebra are:

$$U_z\tilde{\mathcal{G}}(2+1) : \langle J_3, E_1, E_2, P_1, P_2, P_-, P_+ \rangle \equiv \langle -\tilde{J}_3, \tilde{K}_1, \tilde{K}_2, \tilde{P}_1, \tilde{P}_2, \tilde{H}, \tilde{M} \rangle. \quad (3. 10)$$

The Hopf structure of $U_z\tilde{\mathcal{G}}(2+1)$ has the following coproduct and non-vanishing commutation rules:

$$\begin{aligned} \Delta(X) &= 1 \otimes X + X \otimes 1, \quad X \in \{\tilde{M}, \tilde{J}_3, \tilde{K}_1, \tilde{K}_2\}, \\ \Delta(Y) &= e^{-z\tilde{M}} \otimes Y + Y \otimes e^{z\tilde{M}}, \quad Y \in \{\tilde{H}, \tilde{P}_1, \tilde{P}_2\}. \end{aligned} \quad (3. 11)$$

$$\begin{aligned} [\tilde{J}_3, \tilde{K}_i] &= \varepsilon_{ij3} \tilde{K}_j, & [\tilde{J}_3, \tilde{P}_i] &= \varepsilon_{ij3} \tilde{P}_j, \\ [\tilde{K}_i, \tilde{H}] &= \tilde{P}_i, & [\tilde{K}_i, \tilde{P}_j] &= \delta_{ij} \frac{\sinh z\tilde{M}}{z}. \end{aligned} \quad (3. 12)$$

The center of $U_z\tilde{\mathcal{G}}(2+1)$ is generated by the mass \tilde{M} , the energy E_q and the intrinsic angular momentum L_q :

$$E_q^2 = \tilde{P}_1^2 + \tilde{P}_2^2 - \tilde{H} \frac{\sinh z\tilde{M}}{z}, \quad (3. 13)$$

$$L_q = \tilde{K}_1 \tilde{P}_2 - \tilde{K}_2 \tilde{P}_1 - \tilde{J}_3 \frac{\sinh z\tilde{M}}{z}. \quad (3. 14)$$

4 Spin, Hamiltonians and Position Operators

In the classical Poincaré algebra the spin $\vec{\mathcal{S}}$ satisfies the commutation rules:

$$[\mathcal{S}_i, \mathcal{S}_j] = \varepsilon_{ijk} \mathcal{S}_k, \quad [M, \mathcal{S}_i] = 0, \quad i, j, k = 1, 2, 3, \quad (4. 1)$$

where the components \mathcal{S}_i are given in terms of the Pauli–Lubanski vector by

$$M\mathcal{S}_1 = W_{23} + (P_1/P_+)W_+, \quad M\mathcal{S}_2 = W_{13} - (P_2/P_+)W_+, \quad \mathcal{S}_3 = W_+/P_+. \quad (4. 2)$$

The spin $\vec{\mathcal{S}}$ commutes with all the generators of the stability group S_+ except J_3 . In particular, the component \mathcal{S}_3 is a central element of S_+ . Its eigenvalue h is called the null-plane helicity.

Relations (4. 2) enable us to write the Hamiltonians P_- , F_1 and F_2 in terms of the mass and spin operators:

$$\begin{aligned} P_- &= [M^2 + P_1^2 + P_2^2](1/2P_+), \\ F_1 &= [K_3P_1 + E_1P_- - (M\mathcal{S}_2 + P_2\mathcal{S}_3)](1/P_+), \\ F_2 &= [K_3P_2 + E_2P_- - (M\mathcal{S}_1 - P_1\mathcal{S}_3)](1/P_+). \end{aligned} \quad (4. 3)$$

In these expressions the most relevant objects (rather than \mathcal{S}_1 , \mathcal{S}_2 and M) are the products $M\mathcal{S}_1$, $M\mathcal{S}_2$ and M^2 . These quadratic elements are called “reduced Hamiltonians”.

4.1 Quantum Spin and Hamiltonians

The quantum counterpart of this classical structure can be straightforwardly obtained once we know the q -Casimirs of $U_z\mathcal{P}(3+1)$ (3. 7–3. 9). We define the components \mathcal{S}_i^q of the quantum spin in such a way that the following relations hold:

$$M_q\mathcal{S}_1^q = W_{23}^q + \frac{zP_1}{\tanh zP_+}W_+^q, \quad M_q\mathcal{S}_2^q = W_{13}^q - \frac{zP_2}{\tanh zP_+}W_+^q, \quad \mathcal{S}_3^q = \frac{zW_+^q}{\sinh zP_+}. \quad (4. 4)$$

Relations (A.3–A.7) of Appendix A show that the q -spin $\vec{\mathcal{S}}^q$, so introduced, verifies the following commutation rules:

$$[\mathcal{S}_i^q, \mathcal{S}_j^q] = \varepsilon_{ijk}\mathcal{S}_k^q, \quad [M_q, \mathcal{S}_i^q] = 0. \quad (4. 5)$$

Similarly to the classical case, $\vec{\mathcal{S}}^q$ commutes with all the generators of the quantum stability group U_zS_+ except J_3 ; furthermore, the component \mathcal{S}_3^q (quantum helicity) is a central element for U_zS_+ .

The Hamiltonians now adopt the following deformed expressions:

$$\begin{aligned} P_- &= [M_q^2 + P_1^2 + P_2^2] \frac{z}{2 \sinh zP_+}, \\ F_1 &= [K_3P_1 + E_1P_- \cosh zP_+ - (M_q\mathcal{S}_2^q + P_2 \cosh zP_+ \mathcal{S}_3^q)] \frac{z}{\sinh zP_+}, \\ F_2 &= [K_3P_2 + E_2P_- \cosh zP_+ - (M_q\mathcal{S}_1^q - P_1 \cosh zP_+ \mathcal{S}_3^q)] \frac{z}{\sinh zP_+}, \end{aligned} \quad (4. 6)$$

and we call the products $M_q\mathcal{S}_1^q$, $M_q\mathcal{S}_2^q$ and M_q^2 quantum reduced Hamiltonians. Note that in the quantum case $[F_1, F_2] \neq 0$.

A quantum differential realization for $U_z\mathcal{P}(3+1)$ can be obtained in the momentum representation with coordinates $\mathbf{p} = (p_+, p_1, p_2) = (p_+, p_T)$ and $p_+ > 0$. The simplest procedure to derive it consists in getting firstly a realization for the quantum stability group U_zS_+ . It is easy to check that it reads

$$\begin{aligned} P_+ &= p_+, & P_i &= p_i, & E_i &= \frac{\sinh zp_+}{z} \partial_i, \\ J_3 &= p_1 \partial_2 - p_2 \partial_1 + \mathcal{S}_3^q, & K_3 &= \frac{\sinh zp_+}{z} \partial_+, & i &= 1, 2, \end{aligned} \quad (4. 7)$$

where $\partial_i = \frac{\partial}{\partial p_i}$. By introducing (4. 7) in the expressions of the quantum Hamiltonians (4. 6) we get

$$\begin{aligned} P_- &= \frac{z(M_q^2 + p_T^2)}{2 \sinh zp_+}, \\ F_1 &= p_1 \partial_+ + \frac{z(M_q^2 + p_T^2)}{2 \tanh zp_+} \partial_1 - \frac{z}{\sinh zp_+} (M_q \mathcal{S}_2^q + p_2 \cosh zp_+ \mathcal{S}_3^q), \\ F_2 &= p_2 \partial_+ + \frac{z(M_q^2 + p_T^2)}{2 \tanh zp_+} \partial_2 - \frac{z}{\sinh zp_+} (M_q \mathcal{S}_1^q - p_1 \cosh zp_+ \mathcal{S}_3^q), \end{aligned} \quad (4. 8)$$

with $p_T^2 = p_1^2 + p_2^2$.

Let us consider a basis vectors $|\mathbf{p}, h\rangle$ for this representation, i.e., $P_l |\mathbf{p}, h\rangle = p_l |\mathbf{p}, h\rangle$, $l = +, 1, 2$, and $S_3 |\mathbf{p}, h\rangle = h |\mathbf{p}, h\rangle$. The action of the remaining operators is given by (4. 7) and (4. 8). The norm of these eigenvectors is

$$\langle p', h | p, h \rangle = (2\pi)^3 2 \frac{\sinh zp_+}{z} \delta(p' - p) \delta_{h', h}. \quad (4. 9)$$

Now, we introduce a quantum inner product of two arbitrary states depending on the three momentum coordinates $\mathbf{p} = (p_+, p_1, p_2)$ and the helicity h (the quantum spin has classical commutation rules):

$$\langle \phi | \psi \rangle = \frac{1}{(2\pi)^3} \sum_h \int \frac{z d^3 \mathbf{p}}{2 \sinh zp_+} \phi_h^*(\mathbf{p}) \psi_h(\mathbf{p}), \quad (4. 10)$$

where $\phi_h(\mathbf{p}) = \langle p, h | \phi \rangle$. Note that in the limit $z \rightarrow 0$ we recover the appropriate classical inner product. According to expression (4. 10), generators P_+ , and P_l are hermitian operators while K_3 , J_3 and E_l are skew-hermitian. Therefore,

$$\hat{P}_+ \equiv P_+, \quad \hat{P}_l \equiv P_l, \quad \hat{K}_3 \equiv iK_3, \quad \hat{E}_l \equiv iE_l, \quad \hat{J}_3 \equiv iJ_3, \quad (l = 1, 2), \quad (4. 11)$$

gives a hermitian representation of U_zS_+ relative to (4. 10) which induces a hermitian representation of whole q -algebra $U_z(\mathcal{P}(3+1))$. Note that the commutations rules for the hermitian operators (4. 11) are also given by (3.3) by adding the imaginary unit i in the r.h.s of the corresponding commutators. Only P_- out of the remaining three generators is hermitian. On the other hand, let us also remark that in order to extend the $*$ -operation at the Hopf algebra level z has to be consider a dimensional real parameter $[z] = [p_+]^{-1}$ (see coproduct (3.2)).

We can consider the coordinate x^- as a “time” (τ), and the complete wave function will be τ -dependent $f_h(\mathbf{p}, \tau)$. The Hamiltonian $\hat{P}_- \equiv P_-$ provides the evolution of the null-plane in terms of a q -deformed Schrödinger equation $i\partial_\tau f_h(\mathbf{p}, \tau) = \hat{P}_- f_h(\mathbf{p}, \tau)$. From the representation (4. 8) we obtain

$$i\partial_\tau \psi_h(\mathbf{p}, \tau) = \frac{z(m_q^2 + p_T^2)}{2 \sinh zp_+} \psi_h(\mathbf{p}, \tau), \quad (4. 12)$$

where the functions $\psi_h(\mathbf{p}, \tau)$ are square integrable under (4. 10)

$$\frac{1}{(2\pi)^3} \sum_h \int \frac{z d^3 \mathbf{p}}{2 \sinh zp_+} |\psi_h(\mathbf{p}, \tau)|^2 < +\infty. \quad (4. 13)$$

The r.h.s. of this q -Schrödinger equation can be seen as a deformation of the kinetic term $H_0 = (m_q^2 + p_T^2)/2p_+$ of the null-plane bound state equation in quantum chromodynamics [4, 6]. In fact, when z is small enough, we can write

$$\hat{P}_- = \frac{z(m_q^2 + p_T^2)}{2 \sinh zp_+} = \frac{m_q^2 + p_T^2}{2p_+} - \frac{1}{12} z^2 p_+ (m_q^2 + p_T^2) + o(z^4) = H_0 + V_z, \quad (4. 14)$$

and the null-plane deformation intrinsically includes a dynamical part V_z to be computed together with the quantum chromodynamics interaction.

4.2 Quantum Position Operators

In the classical case there exist two null-plane (or mean transverse) position operators \hat{Q}_i ($i = 1, 2$) (see [2]) which measure the position of the center of the longitudinal momentum. Explicitly,

$$\hat{Q}_i = \hat{E}_i / \hat{P}_+, \quad i = 1, 2; \quad (4. 15)$$

and they satisfy the following classical commutation rules with the generators of the stability group S_+ :

$$\begin{aligned} [\hat{Q}_i, \hat{E}_j] &= 0, & [\hat{Q}_i, \hat{K}_3] &= 0, & [\hat{Q}_i, \hat{J}_3] &= i\varepsilon_{ij3} \hat{Q}_j, \\ [\hat{Q}_i, \hat{P}_+] &= 0, & [\hat{Q}_i, \hat{P}_j] &= i\delta_{ij}, & [\hat{Q}_1, \hat{Q}_2] &= 0. \end{aligned} \quad (4. 16)$$

From the definition of \hat{Q}_i (4. 15) in the classical version of the representation (4. 7) the position operators are $\hat{Q}_i = i\frac{\partial}{\partial p_i}$. They commute with the reduced Hamiltonians M^2 , $M\mathcal{S}_i$ and with the helicity \mathcal{S}_3 . The transverse velocity operator ($\frac{d}{d\tau}\hat{Q}_i$) is given by the commutator of \hat{Q}_i with the Hamiltonian \hat{P}_- :

$$\frac{d}{d\tau} \hat{Q}_i = [\hat{Q}_i, \hat{P}_-] = i\hat{P}_i / \hat{P}_+. \quad (4. 17)$$

In order to find quantum deformations for the position operators, \hat{Q}_i , we make the following Ansatz

$$\hat{Q}_i = \hat{E}_i / f(z, \hat{P}_+), \quad i = 1, 2, \quad (4. 18)$$

where $f(z, \hat{P}_+)$ is an arbitrary (smooth) function such that $\lim_{z \rightarrow 0} f(z, \hat{P}_+) = \hat{P}_+$ and $f(z, p_+) \neq 0$ for z real and $p_+ > 0$. Commutators (4. 16) and (4. 17) become in the quantum case (we omit the arguments in the function f):

$$\begin{aligned} [\hat{Q}_i, \hat{E}_j] &= 0, \quad [\hat{Q}_i, \hat{K}_3] = i\hat{E}_i \left\{ \frac{f'}{f^2} \frac{\sinh z\hat{P}_+}{z} - \frac{\cosh z\hat{P}_+}{f} \right\}, \quad [\hat{Q}_i, \hat{J}_3] = i\varepsilon_{ij3}\hat{Q}_j, \\ [\hat{Q}_i, \hat{P}_+] &= 0, \quad [\hat{Q}_i, \hat{P}_j] = i\delta_{ij} \frac{\sinh z\hat{P}_+}{z} \frac{1}{f}, \quad [\hat{Q}_1, \hat{Q}_2] = 0, \quad [\hat{Q}_i, \hat{P}_-] = i\frac{\hat{P}_i}{f}, \end{aligned} \quad (4. 19)$$

where f' is “the formal derivative” of f with respect to \hat{P}_+ . Moreover, whatever the choice of the function f be, the operators \hat{Q}_i are hermitian with respect to (4. 10), and they commute with the quantum reduced Hamiltonians and with the quantum helicity –similarly to the classical case. We analyze the most natural possibilities:

(1) $f = \sinh z\hat{P}_+/z$. The only deformed bracket in (4. 19) is the one corresponding to the transverse velocity operator, i.e., $[\hat{Q}_i, \hat{P}_-] = iz\hat{P}_i/\sinh z\hat{P}_+$. Thus, the quantum realization for \hat{Q}_i is

$$\hat{Q}_i = \frac{z\hat{E}_i}{\sinh z\hat{P}_+}, \quad (4. 20)$$

which, in terms of the differential realization (4. 7) provides the undeformed result $\hat{Q}_i \equiv i\frac{\partial}{\partial p_i}$ for the position operators. As a result, the “generalized” uncertainty principle derived from (4. 20) is given by

$$\Delta\hat{Q}_i\Delta\hat{P}_i \geq \frac{1}{2}, \quad (4. 21)$$

(in this case $[\hat{Q}_i, \hat{P}_j] = i\delta_{ij}$, since we have taken $\hbar = c = 1$).

(2) $f = \tanh z\hat{P}_+/z$. Now the commutation rules (4. 19) read:

$$\begin{aligned} [\hat{Q}_i, \hat{E}_j] &= 0, \quad [\hat{Q}_i, \hat{K}_3] = i\hat{E}_i z \sinh z\hat{P}_+, \quad [\hat{Q}_i, \hat{J}_3] = i\varepsilon_{ij3}\hat{Q}_j, \\ [\hat{Q}_i, \hat{P}_+] &= 0, \quad [\hat{Q}_i, \hat{P}_j] = i\delta_{ij} \cosh z\hat{P}_+, \quad [\hat{Q}_1, \hat{Q}_2] = 0. \end{aligned} \quad (4. 22)$$

The time derivative of \hat{Q}_i turns into

$$\frac{d}{d\tau}\hat{Q}_i = [\hat{Q}_i, \hat{P}_-] = i\frac{z\hat{P}_i}{\tanh z\hat{P}_+}, \quad (4. 23)$$

and the corresponding generalized uncertainty relation is given by

$$\Delta\hat{Q}_i\Delta\hat{P}_i \geq \frac{1}{2}\Delta(\cosh z\hat{P}_+). \quad (4. 24)$$

A power series expansion gives

$$\Delta\hat{Q}_i\Delta\hat{P}_i \geq \frac{1}{2}(1 + \frac{z^2}{2}(\Delta\hat{P}_+)^2 + \dots), \quad (4. 25)$$

a result similar to the one presented in [13] that concerns a generalized uncertainty principle in Quantum Gravity. In our case the “classical” value $\Delta\hat{Q}_i\Delta\hat{P}_i \geq \frac{1}{2}$ appears deformed by addition of terms depending on $\Delta\hat{P}_+$.

5 Concluding remarks

We have presented a new quantum deformation of the algebra $\mathcal{P}(3+1)$ whose natural framework is the null-plane basis for this algebra. From the point of view of quantum group theory, this deformation presents some interesting properties that distinguish it from the other ones already known. The essential point is that the classical r -matrix (3. 5) is triangular; thus, a $*_{\hbar}$ -product quantizing the algebra of representative functions on the group exists. First order information concerning the corresponding null-plane quantum Poincaré group can be extracted from the Lie bialgebra structure (3. 6). In particular, for the (now non-commutative) coordinates $(\hat{p}_1, \hat{p}_2, \hat{p}_+)$ we shall have

$$[\hat{p}_1, \hat{p}_+] = 2z\hat{p}_1 + o(z^2), \quad [\hat{p}_2, \hat{p}_+] = 2z\hat{p}_2 + o(z^2). \quad (5. 1)$$

Moreover, since r is a solution of the CYBE, a universal R -matrix turning this non-standard deformation into a triangular Hopf algebra could be hopefully derived. All these formal aspects deserve further study.

From a physical point of view, the most relevant feature of the results here displayed is the parallelism between the classical subalgebras and the Hopf algebra ones induced after deformation. This fact is the cornerstone behind the straightforward generalization of the classical constructions intrinsically containing a modification of both the kinematical and dynamical classical symmetries. In this way, by means of a hermitian representation of the quantum isotopy algebra of the null-plane Π_n^0 , it has been possible to define a family of (hermitian) Newton–Wigner position operators corresponding to the transverse coordinates q_1 and q_2 . These operators are not univocally defined as there exists some freedom in the choice of a function $f(z, \hat{P}_+)$ given in the above section. The Lie bracket between the position and the momentum operators gives rise to a q -deformed generalized uncertainty principle whose expression is

$$\Delta\hat{Q}_i\Delta\hat{P}_i \geq \frac{1}{2}\Delta\left(\frac{\sinh z\hat{P}_+}{zf(z, \hat{P}_+)}\right). \quad (5. 2)$$

Note that expressions (4. 21) and (4. 24) are particular cases of (5.2) according to the different choices of f .

The time evolution of a null-plane deformed system (given by the q -Schrödinger equation (4. 12)) is also a consequence of the existence of an extended (2+1) Galilean Hopf subalgebra, in which \hat{P}_- plays the role of the deformed Hamiltonian and \hat{P}_+ is the mass operator in perfect agreement with the classical version. Quantum deformed reformulations of classical models based in the null-plane description (and in particular, the role of the deformed coproduct for composite systems) seem to be worth studying.

Appendix A: Commutation relations for the classical and quantum Pauli–Lubanski operator

The non-vanishing commutation relations of the components W_i of the Pauli–Lubanski vector (2. 7) among themselves and with the generators of $\mathcal{P}(3+1)$ are:

$$\begin{aligned} [J_3, W_{13}] &= -W_{23}, & [E_2, W_{13}] &= W_+, & [F_2, W_{13}] &= -W_-, \\ [J_3, W_{23}] &= W_{13}, & [E_1, W_{23}] &= -W_+, & [F_1, W_{23}] &= W_-, \\ [E_1, W_-] &= W_{23}, & [E_2, W_-] &= -W_{13}, & [K_3, W_-] &= -W_-, \\ [F_1, W_+] &= -W_{23}, & [F_2, W_+] &= W_{13}, & [K_3, W_+] &= W_+; \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} [W_{13}, W_+] &= W_+ P_1 + W_{23} P_+, & [W_{13}, W_-] &= -W_- P_1 + W_{23} P_-, \\ [W_{23}, W_+] &= W_+ P_2 - W_{13} P_+, & [W_{23}, W_-] &= -W_- P_2 - W_{13} P_-, \\ [W_{23}, W_{13}] &= W_+ P_- + W_- P_+, & [W_+, W_-] &= W_{13} P_1 + W_{23} P_2. \end{aligned} \quad (\text{A.2})$$

The quantum analogues of the above relations, which allow to prove that W_q^2 (3. 8) belongs to the center of $U_z \mathcal{P}(3+1)$, are:

$$\begin{aligned} [J_3, W_{13}^q] &= -W_{23}^q, & [F_1, W_{13}^q] &= z^2 W_+^q P_1 P_2, \\ [E_2, W_{13}^q] &= W_+^q \cosh z P_+, & [K_3, W_{13}^q] &= z W_+^q P_2 \sinh z P_+, \\ [F_2, W_{13}^q] &= -W_-^q \cosh z P_+ + z^2 W_+^q \{M_q^2 + P_2^2\}, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} [J_3, W_{23}^q] &= W_{13}^q, & [F_2, W_{23}^q] &= -z^2 W_+^q P_1 P_2, \\ [E_1, W_{23}^q] &= -W_+^q \cosh z P_+, & [K_3, W_{23}^q] &= -z W_+^q P_1 \sinh z P_+, \\ [F_1, W_{23}^q] &= W_-^q \cosh z P_+ - z^2 W_+^q \{M_q^2 + P_1^2\}, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} [E_1, W_-^q] &= W_{23}^q, & [F_1, W_-^q] &= -z^2 W_+^q P_1 P_-, \\ [E_2, W_-^q] &= -W_{13}^q, & [F_2, W_-^q] &= -z^2 W_+^q P_2 P_-, \end{aligned} \quad (\text{A.5})$$

$$[K_3, W_-^q] = -W_-^q \cosh z P_+ + z^2 W_+^q \left\{ M_q^2 - P_- \frac{\sinh z P_+}{z} \right\},$$

$$[F_1, W_+^q] = -W_{23}^q, \quad [F_2, W_+^q] = W_{13}^q, \quad [K_3, W_+^q] = W_+^q \cosh z P_+; \quad (\text{A.6})$$

$$\begin{aligned} [W_{13}^q, W_+^q] &= W_+^q P_1 \cosh z P_+ + W_{23}^q \frac{\sinh z P_+}{z}, \\ [W_{23}^q, W_+^q] &= W_+^q P_2 \cosh z P_+ - W_{13}^q \frac{\sinh z P_+}{z}, \\ [W_{23}^q, W_{13}^q] &= W_+^q \{P_- \cosh^2 z P_+ - z M_q^2 \sinh z P_+\} + W_-^q \frac{\sinh 2z P_+}{2z}, \\ [W_{13}^q, W_-^q] &= -W_-^q P_1 \cosh z P_+ + W_{23}^q P_- \cosh z P_+ + z^2 M_q^2 W_+^q, \\ [W_{23}^q, W_-^q] &= -W_-^q P_2 \cosh z P_+ - W_{13}^q P_- \cosh z P_+ + z^2 M_q^2 W_+^q, \\ [W_+^q, W_-^q] &= W_{13}^q P_1 + W_{23}^q P_2. \end{aligned} \quad (\text{A.7})$$

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